

NASA/TP—2001-210515



# Stress Formulation in Three-Dimensional Elasticity

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September 2001

This report is a formal draft or working paper, intended to solicit comments and ideas from a technical peer group.

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# Stress Formulation in Three-Dimensional Elasticity

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## Summary

The theory of elasticity evolved over centuries through the contributions of eminent scientists like Cauchy, Navier, Hooke, Saint Venant, and others. It was deemed complete when Saint Venant provided the strain formulation in 1860. However, unlike Cauchy, who addressed equilibrium in the field and on the boundary, the strain formulation was confined only to the field. Saint Venant overlooked the compatibility on the boundary. Because of this deficiency, a direct stress formulation could not be developed. Stress with traditional methods must be recovered by backcalculation: differentiating either the displacement or the stress function. We have addressed the compatibility on the boundary. Augmentation of these conditions has completed the stress formulation in elasticity, opening up a way for a direct determination of stress without the intermediate step of calculating the displacement or the stress function. This Completed Beltrami-Michell Formulation (CBMF) can be specialized to derive the traditional methods, but the reverse is not possible. Elasticity solutions must be verified for the compliance of the new equation because the boundary compatibility conditions expressed in terms of displacement are not trivially satisfied. This paper presents the variational derivation of the stress formulation, illustrates the method, examines attributes and benefits, and outlines the future course of research.

## 1.0 Introduction

The theory of elasticity evolved over centuries through the contributions of eminent scientists like Cauchy, Navier, Hooke, Saint Venant, and others (see fig. 1). Saint Venant, through his strain formulation, presumed to have provided the last set of equations in 1860. Decades earlier, Cauchy had formulated the equilibrium in the field and on the boundary of an elastic continuum. The equilibrium and compatibility formulations were deemed complete even before the turn of the 20th century. This presumption is in error. Saint Venant, credited for the field equations, overlooked the boundary formulation. In other words, the theory of elasticity camouflaged a deficiency in the compatibility formulation for well over a century. This weakness diverted the development of a direct stress determination method. The traditional methods recovered stress by backcalculations: either by differentiating the displacement or the stress function. We have addressed the boundary compatibility condition (BCC). Augmenting available elasticity equations with the BCC has led to the completion of the Beltrami-Michell formulation (CBMF). This method with stress as the primary unknown can solve displacement and mixed boundary value problems in elasticity. A direct stress determination method, like the CBMF, however was envisioned by Beltrami and Michell (ref. 1) following the strain formulation in 1860. Michell's thought is expressed by Love (ref. 2) in the following quotation:

“It is possible by taking account of these relations [*the compatibility conditions*] to obtain a complete system of equations which must be satisfied by stress components, and thus the way is open for a direct determination of stress without the intermediate steps of forming and solving differential equations to determine the components of displacements.”

Their method, or the classical Beltrami-Michell formulation, was incomplete because it did not include our boundary conditions. It thus has limited application. The completed method, or the CBMF, is the versatile elasticity formulation. The CBMF can be specialized to obtain Navier's displacement method and Airy's stress function formulation. The CBMF cannot be derived from the later two methods. Elasticity solutions that have been obtained by the two traditional methods must be verified for the compliance of the new equations because the boundary compatibility conditions expressed in terms of the displacement functions are not trivially satisfied. This paper presents the CBMF for three-dimensional elasticity in the

subsequent sections: completed Beltrami-Michell stress formulation, properties of the compatibility conditions, illustrative examples, a discussion on attributes and benefits, future course of research, and conclusions.

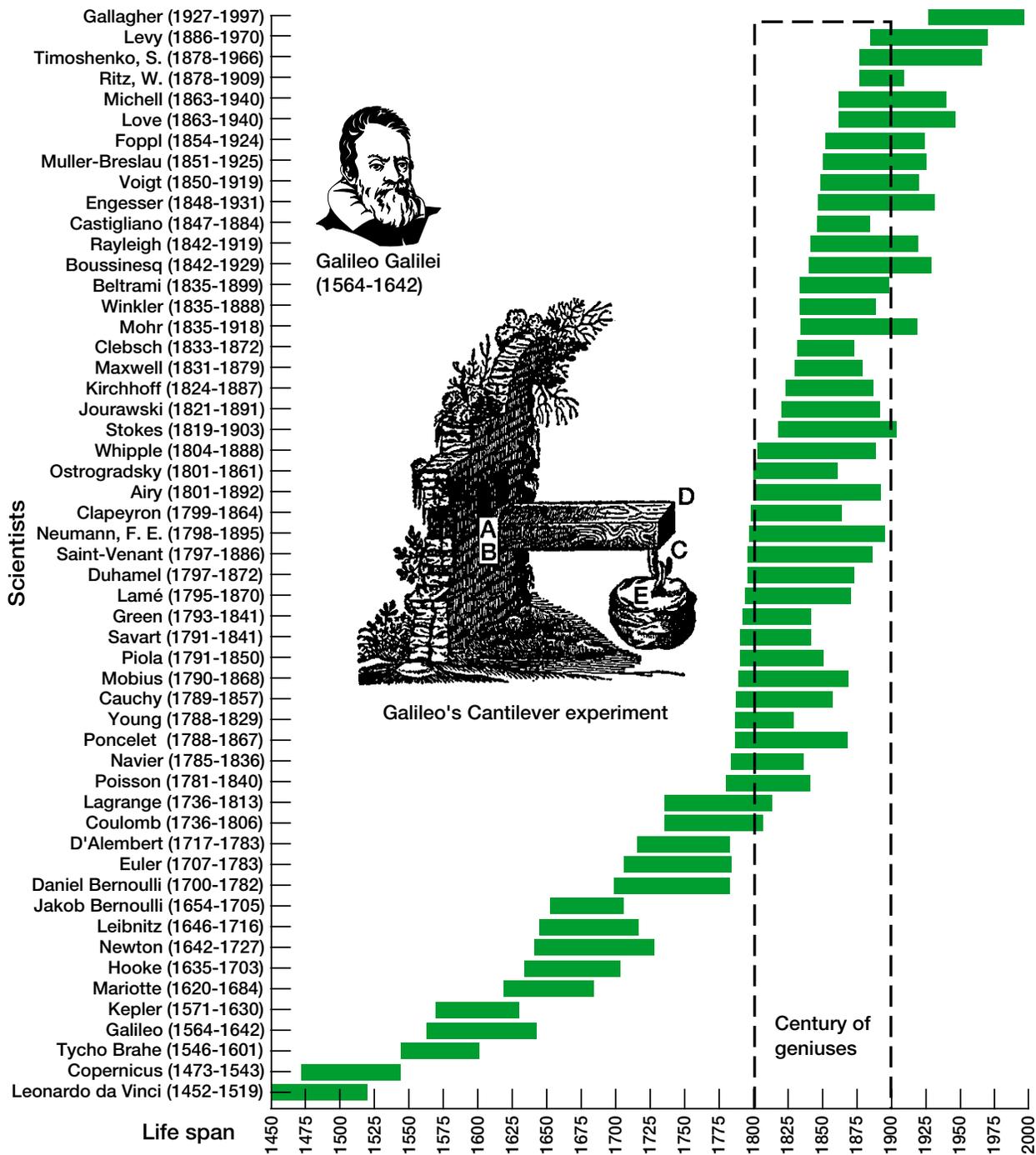


Figure 1.—The theory of structures had camouflaged a deficiency in the compatibility formulation since 1860.

## 2.0 Completed Beltrami-Michell Stress Formulation

We emphasize that the CBMF can be derived only through a variational approach. The new equations, the boundary compatibility conditions, are not amenable to a direct derivation. The CBMF is obtained from the stationary condition of the variational functional (ref. 3) of the Integrated Force Method (IFM). The stationary condition yields the field equations and the boundary conditions as well as the displacement continuity conditions. The functional  $\pi_s$  has three terms— $A$ ,  $B$ , and  $W$ —as follows:

$$\pi_s = A + B - W \quad (1)$$

Term  $A$  represents the internal energy expressed in terms of stress ( $\sigma$ ) and displacement ( $u, v, w$ ) as follows:

$$A = \int_V \left\{ \sigma_x \frac{\partial u}{\partial x} + \sigma_y \frac{\partial v}{\partial y} + \sigma_z \frac{\partial w}{\partial z} + \tau_{xy} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + \tau_{yz} \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) + \tau_{zx} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \right\} dV \quad (2a)$$

Term  $B$  represents the complementary internal energy. It is expressed in strain ( $\epsilon$ ) and the stress function ( $\phi$ ) as

$$B = \int_V \left\{ \epsilon_x \left( \frac{\partial^2 \phi_3}{\partial y^2} + \frac{\partial^2 \phi_2}{\partial z^2} \right) + \epsilon_y \left( \frac{\partial^2 \phi_1}{\partial z^2} + \frac{\partial^2 \phi_3}{\partial x^2} \right) + \epsilon_z \left( \frac{\partial^2 \phi_2}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2} \right) + \gamma_{xy} \left( -\frac{\partial^2 \phi_3}{\partial x \partial y} \right) + \gamma_{yz} \left( -\frac{\partial^2 \phi_1}{\partial y \partial z} \right) + \gamma_{zx} \left( -\frac{\partial^2 \phi_2}{\partial z \partial x} \right) \right\} dV \quad (2b)$$

The term  $W$  represents the potential of the work done due to the prescribed traction on surface  $\Gamma^s$ , the prescribed displacements on  $\Gamma^u$ , and the specified body forces in the volume  $V$  of the continuum (see fig. 2) as

$$W = \int_{\Gamma^s} (\bar{P}_x u + \bar{P}_y v + \bar{P}_z w) dS + \int_{\Gamma^u} (P_x \bar{u} + P_y \bar{v} + P_z \bar{w}) dS + \int_V (B_x u + B_y v + B_z w) dV \quad (2c)$$

where  $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz},$  and  $\tau_{zx}$  are the six stress components;  $\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{xy}, \gamma_{yz},$  and  $\gamma_{zx}$  are the six strain components;  $u, v,$  and  $w$  are the three displacement components;  $\phi_1, \phi_2,$  and  $\phi_3$  are the three stress functions;  $\bar{P}_x, \bar{P}_y,$  and  $\bar{P}_z$  are the three prescribed tractions;  $P_x, P_y,$  and  $P_z$  are reactions where displacements  $\bar{u}, \bar{v},$  and  $\bar{w}$  are prescribed; and  $B_x, B_y,$  and  $B_z$  are the three body force components.

The three stress functions are defined as

$$\begin{aligned} \sigma_x &= \frac{\partial^2 \phi_3}{\partial y^2} + \frac{\partial^2 \phi_2}{\partial z^2} & \sigma_y &= \frac{\partial^2 \phi_1}{\partial z^2} + \frac{\partial^2 \phi_3}{\partial x^2} & \sigma_z &= \frac{\partial^2 \phi_2}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2} \\ \tau_{xy} &= -\frac{\partial^2 \phi_3}{\partial x \partial y} & \tau_{yz} &= -\frac{\partial^2 \phi_1}{\partial y \partial z} & \tau_{zx} &= -\frac{\partial^2 \phi_2}{\partial z \partial x} \end{aligned} \quad (3)$$

The stationary condition of  $\pi_s$  with respect to the displacements and stress functions yields all the equations of the CBMF as follows:

$$\begin{aligned}
\delta\pi_s = & \left[ -\int_V \left\{ \frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z} + B_x \right\} \delta u + \left\{ \frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{yz}}{\partial z} + B_y \right\} \delta v + \right. \\
& \left. \left\{ \frac{\partial\tau_{zx}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\sigma_z}{\partial z} + B_z \right\} \delta w + \left\{ \frac{\partial^2\varepsilon_y}{\partial z^2} + \frac{\partial^2\varepsilon_z}{\partial y^2} - \frac{\partial^2\gamma_{yz}}{\partial z\partial y} \right\} \delta\phi_1 + \right. \\
& \left. \left\{ \frac{\partial^2\varepsilon_x}{\partial z^2} + \frac{\partial^2\varepsilon_z}{\partial x^2} - \frac{\partial^2\gamma_{xz}}{\partial z\partial x} \right\} \delta\phi_2 + \left\{ \frac{\partial^2\varepsilon_x}{\partial y^2} + \frac{\partial^2\varepsilon_x}{\partial x^2} - \frac{\partial^2\gamma_{xy}}{\partial x\partial y} \right\} \delta\phi_3 \right] dV + \\
& \int_S \left[ \left\{ a_{vx}\sigma_x + a_{vy}\tau_{xy} + a_{vz}\tau_{zx} - \bar{P}_x \right\} \delta u + \left\{ a_{vx}\tau_{xy} + a_{vy}\sigma_y + a_{vz}\tau_{yz} - \bar{P}_y \right\} \delta v + \right. \\
& \left. \left\{ a_{vx}\tau_{zx} + a_{vy}\tau_{yz} + a_{vz}\sigma_z - \bar{P}_z \right\} \delta w - \left\{ \frac{\partial}{\partial z} \left( a_{vz}\varepsilon_y - \frac{a_{vy}}{2}\gamma_{yz} \right) + \frac{\partial}{\partial y} \left( a_{vy}\varepsilon_z - \frac{a_{vz}}{2}\gamma_{yz} \right) \right\} \delta\phi_1 - \right. \\
& \left. \left\{ \frac{\partial}{\partial x} \left( a_{vx}\varepsilon_z - \frac{a_{vz}}{2}\gamma_{zx} \right) + \frac{\partial}{\partial z} \left( a_{vz}\varepsilon_x - \frac{a_{vx}}{2}\gamma_{xz} \right) \right\} \delta\phi_2 - \right. \\
& \left. \left\{ \frac{\partial}{\partial y} \left( a_{vy}\varepsilon_x - \frac{a_{vx}}{2}\gamma_{xy} \right) + \frac{\partial}{\partial x} \left( a_{vx}\varepsilon_y - \frac{a_{vy}}{2}\gamma_{xy} \right) \right\} \delta\phi_3 \right] dS + \\
& \int_S \left[ (u - \bar{u}) \delta (a_{vx}\sigma_x + a_{vy}\tau_{xy} + a_{vz}\tau_{zx}) + (v - \bar{v}) \delta (a_{vx}\tau_{xy} + a_{vy}\sigma_y + a_{vz}\tau_{yz}) + \right. \\
& \left. (w - \bar{w}) \delta (a_{vx}\tau_{zx} + a_{vy}\tau_{yz} + a_{vz}\sigma_z) \right] dS \tag{4}
\end{aligned}$$

## 2.1 Field Equilibrium Equations

The three equilibrium equations in the field are obtained as the coefficients of the three incremental displacement components ( $\delta u$ ,  $\delta v$ ,  $\delta w$ ) in eq. (4).

$$\begin{aligned}
\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z} + B_x &= 0 \\
\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{yz}}{\partial z} + B_y &= 0 \\
\frac{\partial\tau_{zx}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\sigma_z}{\partial z} + B_z &= 0
\end{aligned} \tag{5}$$

The formulation of the field equilibrium equations (5) is credited to Cauchy (see fig. 1).

## 2.2 Field Compatibility Conditions

The three CC in the field are obtained as the coefficients of the three incremental stress (see fig. 1) functions ( $\delta\phi_1$ ,  $\delta\phi_2$ ,  $\delta\phi_3$ ) in eq. (4). The strain CC in the field, referred to as the “strain formulation,” is credited to Saint Venant. For an isotropic material with Young’s modulus  $E$  and Poisson’s ratio  $\nu$ , the CC expressed in terms of stress are as follows:

$$\begin{aligned} \frac{\partial^2}{\partial z^2} (\sigma_y - \nu\sigma_z - \nu\sigma_x) + \frac{\partial^2}{\partial y^2} (\sigma_z - \nu\sigma_x - \nu\sigma_y) - 2(1+\nu) \frac{\partial^2 \tau_{yz}}{\partial y \partial z} &= 0 \\ \frac{\partial^2}{\partial x^2} (\sigma_z - \nu\sigma_x - \nu\sigma_y) + \frac{\partial^2}{\partial z^2} (\sigma_x - \nu\sigma_y - \nu\sigma_z) - 2(1+\nu) \frac{\partial^2 \tau_{zx}}{\partial z \partial x} &= 0 \\ \frac{\partial^2}{\partial y^2} (\sigma_x - \nu\sigma_y - \nu\sigma_z) + \frac{\partial^2}{\partial x^2} (\sigma_y - \nu\sigma_z - \nu\sigma_x) - 2(1+\nu) \frac{\partial^2 \tau_{xy}}{\partial x \partial y} &= 0 \end{aligned} \quad (6)$$

The field CC given by eq. (6) can be further simplified to obtain the canonical form given in reference 4.

## 2.3 Boundary Equilibrium or Traction Conditions

The three equilibrium equations on the boundary are obtained as the coefficients of the variational displacements ( $\delta u$ ,  $\delta v$ ,  $\delta w$ ) in the surface integral terms in eq. (4):

$$\begin{aligned} a_{vx} \sigma_x + a_{vy} \tau_{xy} + a_{vz} \tau_{xz} &= \bar{P}_x \\ a_{vx} \tau_{xy} + a_{vy} \sigma_y + a_{vz} \tau_{xz} &= \bar{P}_y \\ \tau_{xz} a_{vx} + \tau_{yz} a_{vy} + \sigma_y a_{vz} &= \bar{P}_y \end{aligned} \quad (7)$$

The traction conditions credited to Cauchy are indeterminate because six stresses are expressed in terms of three equations

## 2.4 Boundary Compatibility Conditions

The BCC are obtained as the coefficients of the variational terms ( $\delta\phi_1$ ,  $\delta\phi_2$ ,  $\delta\phi_3$ ) in the surface integrals in eq. (4). For an isotropic material, the BCC when expressed in stress become

$$\begin{aligned} \frac{\partial}{\partial z} \{ a_{vz} (\sigma_y - \nu\sigma_z - \nu\sigma_x) - a_{vy} (1+\nu) \tau_{yz} \} + \frac{\partial}{\partial y} \{ a_{vy} (\sigma_z - \nu\sigma_x - \nu\sigma_y) - a_{vz} (1+\nu) \tau_{yz} \} &= 0 \\ \frac{\partial}{\partial x} \{ a_{vx} (\sigma_z - \nu\sigma_x - \nu\sigma_y) - a_{vz} (1+\nu) \tau_{zx} \} + \frac{\partial}{\partial z} \{ a_{vz} (\sigma_x - \nu\sigma_y - \nu\sigma_z) - a_{vx} (1+\nu) \tau_{zx} \} &= 0 \\ \frac{\partial}{\partial y} \{ a_{vy} (\sigma_x - \nu\sigma_y - \nu\sigma_z) - a_{vx} (1+\nu) \tau_{xy} \} + \frac{\partial}{\partial x} \{ a_{vx} (\sigma_y - \nu\sigma_z - \nu\sigma_x) - a_{vy} (1+\nu) \tau_{xy} \} &= 0 \end{aligned} \quad (8)$$

The three BCC given by eq. (8) had been missing since 1860. The traction condition given by equation (6) and the BCC ensure stress determinacy on the boundary.

## 2.5 Displacement Continuity Conditions

Displacement boundary conditions are obtained as the coefficients of the variational reaction terms  $\{\delta(a_{vx}\sigma_x + a_{vy}\tau_{xy} + a_{vz}\tau_{zx}), \delta(a_{vx}\tau_{xy} + \dots), \text{ and } \delta(a_{vx}\tau_{zx} + \dots)\}$  in the surface integral in eq. (4):

$$\begin{aligned} u &= \bar{u} \\ v &= \bar{v} \\ w &= \bar{w} \end{aligned} \quad (9)$$

The variational functional of IFM yields almost all the equations of elasticity:

- (1) The EE in the field and on the boundary: These are identical to Cauchy's equations.
- (2) The CC in the field and on the boundary: The field equations are credited to Saint Venant. The BCC are the new equations.
- (3) The displacement continuity conditions: The variational method provides a difficult, but elegant, derivation of the continuity conditions.
- (4) The IFM functional can be specialized to obtain the potential and the complementary energy functionals.

CBMF is defined through the EE and the CC in the field and on the boundary of an elastic continuum, given by eqs. (5) to (8). Stress determination by CBFM does not require the displacement boundary conditions. The displacement functions,  $u(x, y, z)$ ,  $v(x, y, z)$ , and  $w(x, y, z)$ , if required, can be determined by integrating the known stress. The calculation of the constants of integration in the displacement functions requires the displacement boundary conditions given by eq. (9). CBFM can be used for analysis of stress, displacement, and mixed boundary value problems. A correct solution must satisfy all elasticity equations (eqs. (5) to (9)). Solutions obtained without the use of the BCC should be verified for their compliance, especially for a continuum with deformable boundaries. In traditional solutions to the elasticity problem, displacement continuity conditions are augmented with the slope or derivative of displacement. The "slope" condition somewhat resembles the boundary compatibility condition.

## 3.0 Properties of the Compatibility Conditions

Strain or deformation balance is the physical concept behind the controller-type compatibility conditions. In elasticity, for example, the strains ( $\epsilon$ ) are controlled,  $f(\epsilon_x, \epsilon_y, \dots, \gamma_{zx}) = 0$ , or the deformations ( $\beta$ ) are balanced,  $f(\beta_1, \beta_2, \dots, \beta_n) = 0$ , in discrete structural systems. The concept of balancing stress and force on a differential block to generate the EE cannot be applied to generate the CC. The procedure adopted by Sokolnikoff (ref. 4)—the elimination of displacement from the strain displacement relations to derive the field CC—cannot be extended to generate the BCC. Their generation requires the utilization of variational calculus. The BCC derived from the stationary condition of the IFM variational functional can, however, be verified through an application of the theorems of integral calculus.

### 3.1 Verification of the Boundary Compatibility Conditions

Green's theorem can be used to verify the BCC. This theorem in two-dimensions can be written as

$$\int_S \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) ds = \int_{\Gamma} (a_{vx}P + a_{vy}Q) d\ell \quad (10)$$

where  $P$  and  $Q$  are differentiable functions and  $\Gamma$  represents the boundary of the domain  $S$ .

The BCC in two-dimensional elasticity can be written as

$$a_{vx} \left[ \left( \frac{\partial \epsilon_y}{\partial x} \right) - \frac{1}{2} \left( \frac{\partial \gamma_{xy}}{\partial y} \right) \right] + a_{vy} \left[ \left( \frac{\partial \epsilon_x}{\partial y} \right) - \frac{1}{2} \left( \frac{\partial \gamma_{xy}}{\partial x} \right) \right]$$

This condition is recovered in the line integral term in eq. (10) when the field compatibility condition given by

$$\frac{\partial \epsilon_x}{\partial y^2} + \frac{\partial \epsilon_y}{\partial x^2} - \frac{\partial \gamma_{xy}}{\partial x \partial y} = \frac{\partial}{\partial x} \left[ \left( \frac{\partial \epsilon_y}{\partial x} \right) - \frac{1}{2} \left( \frac{\partial \gamma_{xy}}{\partial y} \right) \right] + \frac{\partial}{\partial y} \left[ \left( \frac{\partial \epsilon_x}{\partial y} \right) - \frac{1}{2} \left( \frac{\partial \gamma_{xy}}{\partial x} \right) \right]$$

is substituted in the surface integral. The reader can verify the BCC for three-dimensional elasticity by repeating the procedure.

### 3.2 Nontrivial Property of the Boundary Compatibility Conditions

The compatibility conditions in the field are automatically satisfied when expressed in terms of continuous displacement functions  $(u, v, w)$ , leading to an identity like  $\{f(u, v, w) - f(u, v, w)\} = 0$ . The boundary compatibility conditions expressed in terms of continuous displacements are not satisfied automatically. For three-dimensional elasticity, the three BCC written in terms of displacement take the following forms:

$$\begin{aligned} \mathfrak{R}_1 &= a_{vy} \frac{\partial^2 u}{\partial z^2} + a_{vz} \frac{\partial^2 w}{\partial y^2} - \frac{\partial^2}{\partial y \partial z} (a_{vz} v + a_{vy} w) = 0 \\ \mathfrak{R}_2 &= a_{vz} \frac{\partial^2 w}{\partial x^2} + a_{vx} \frac{\partial^2 u}{\partial z^2} - \frac{\partial^2}{\partial z \partial x} (a_{vx} w + a_{vz} u) = 0 \\ \mathfrak{R}_3 &= a_{vy} \frac{\partial^2 u}{\partial x^2} + a_{vx} \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2}{\partial x \partial y} (a_{vy} u + a_{vx} v) = 0 \end{aligned} \quad (11)$$

For three-dimensional elasticity, the compatibility at a boundary interface, defined by an outward normal  $(\vec{n})$  with the direction cosines  $a_{vx}$ ,  $a_{vy}$ , and  $a_{vz}$ , shown in figure 2, will be satisfied provided the following three residues are matched at the left and right of the boundary.

$$\mathfrak{R}_1^L + \mathfrak{R}_1^R = 0 \quad \mathfrak{R}_2^L + \mathfrak{R}_2^R = 0 \quad \mathfrak{R}_3^L + \mathfrak{R}_3^R = 0 \quad (12)$$

where  $\mathfrak{R}_1^L$  and  $\mathfrak{R}_1^R$  refer to the residues for the displacement functions on the right and left of the boundary  $P-Q$  in figure 2. The compatibility compliance imposes a restriction on the derivatives of the displacement functions.

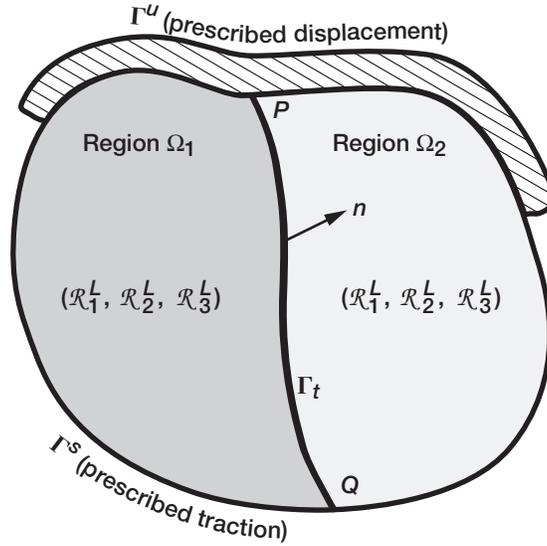


Figure 2.—A composite elastic continuum.

### 3.3 Compliance of the Boundary Compatibility Conditions

The compliance of the BCC for membrane analysis is illustrated considering an example of a simple stiffness-method-based finite element model with a four-node rectangular and a three-node triangular element as shown in figure 3. Along the interface connecting nodes (2 and 4) the compatibility compliance is defined by

$$\mathfrak{R}^{\text{interface}} = \mathfrak{R}^{\text{rectangular}} + \mathfrak{R}^{\text{triangular}} = 0 \quad (13)$$

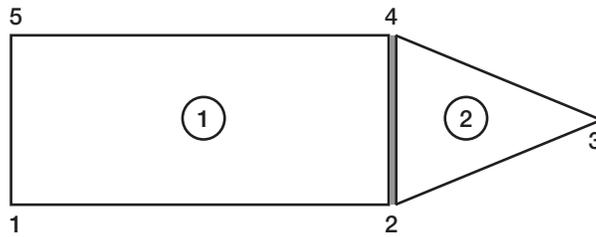


Figure 3.—Boundary compatibility compliance for two-element model.

where the residue function ( $\mathfrak{R}$ ) for each element defined in terms of displacements ( $u, v$ ) is

$$\mathfrak{R} = a_{vx} \left[ \frac{\partial^2 v}{\partial x \partial y} - \frac{1}{2} \left( \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y} \right) \right] + a_{vy} \left[ \frac{\partial^2 u}{\partial x \partial y} - \frac{1}{2} \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} \right) \right] \quad (14)$$

Consider displacement functions for the rectangular membrane elements (ref. 5) as follows:

$${}^{\text{rectangular}}u(x, y) = c_1x + c_2xy + c_3y + c_4 \quad (15)$$

Likewise, the  ${}^{\text{rectangular}}v(x, y)$  displacement function can be defined. The eight constants ( $c_1, c_2, \dots, c_8$ ) can be linked to the eight nodal displacements of the rectangular element.

Displacement function for the triangular element can be written as

$${}^{\text{triangular}}u(x, y) = d_1x + d_2y + d_3 \quad (16)$$

Likewise the ( ${}^{\text{triangular}}v(x, y)$ ) displacement can be written. The six constants ( $d_1, d_2, \dots, d_6$ ) can be linked to the six nodal displacements of the triangular element.

The contribution to the residue function ( $\mathfrak{R}$ ) for each of the two elements can be obtained as

$$\begin{aligned} \mathfrak{R}^{\text{rectangular}(1-2)} &= 0.5 \{a_{vx}c_6 + a_{vy}c_2\} \\ \mathfrak{R}^{\text{triangular}(1-2)} &= 0 \end{aligned} \quad (17)$$

The boundary compliance ( $\mathfrak{R}^{\text{rectangular}} + \mathfrak{R}^{\text{triangular}} = 0$ ) at the interface of the finite element model shown in figure 3 is not satisfied because  $\mathfrak{R}^{\text{interface}} = 0.5\{a_{vx}c_6 + a_{vy}c_2\}$  and  $c_6 = c_2 \neq 0$ . In finite element analysis, the traditional assumption that the stiffness method satisfies the compatibility conditions a priori needs to be reviewed with respect to compliance of the boundary compatibility conditions.

### 3.4 Attributes of Elasticity Operators

The equilibrium operator and the compatibility operator are related. The operation of the field equilibrium operator [ $\mathcal{L}_\epsilon$ ] on the field compatibility operator [ $\mathcal{L}_c$ ] produces a null condition:

$$[\mathcal{L}_\epsilon] [\mathcal{L}_c]^T = \{0\} \quad (18a)$$

The field equilibrium equations, in operator form, can be written as

$$\begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \tau_{xy} \\ \sigma_y \end{Bmatrix} = \begin{Bmatrix} -B_x \\ -B_y \end{Bmatrix} \quad \text{or} \quad \mathcal{L}_\epsilon \{\sigma\} = -\{B\} \quad (18b)$$

Likewise, the strain compatibility conditions in the field can be represented as

$$\begin{bmatrix} \frac{\partial^2}{\partial y^2} & -\frac{\partial^2}{\partial x \partial y} & \frac{\partial^2}{\partial x^2} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \gamma_{xy} \\ \epsilon_y \end{Bmatrix} = \{0\} \quad \text{or} \quad \mathcal{L}_c \{\epsilon\} = 0 \quad (18c)$$

Equation (18a) can be verified by direct substitution. The equilibrium and strain displacement operators also are the transpose of each other.

$$[\mathcal{L}_\varepsilon] = [\mathcal{L}]_{\text{sdr}}^T \quad (18d)$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \quad \text{or} \quad \{\varepsilon\} = [\mathcal{L}_{\text{sdr}}] \begin{Bmatrix} u \\ v \end{Bmatrix}$$

The strain displacement relation retains the genus of the equilibrium and compatibility concepts. Both the field CC and the field EE can be derived from the strain displacement relations by eliminating the displacements and applying virtual work principles, respectively. The null property given by equations (18a) and (18b) shows the relationships that exist between the equilibrium, compatibility, and strain displacement concepts. The equilibrium and the compatibility concept remain the same either in the field or on the boundary. Their equation forms differ to accommodate the field or boundary.

#### 4.0 Illustrative Example: Composite Circular Plate Under Thermomechanical Loads

The CBMF is illustrated through the analysis of a composite plate. The moment and displacement of the problem are functions of a single radial coordinate. This trivial problem, solved in closed form, has identical solutions via CBMF and Navier's displacement method, because for the radial symmetric plate the boundary compatibility condition reduces to the slope continuity condition. The problem, however, illustrates the treatment of the boundary compatibility condition and the displacement boundary condition in the stress formulation.

The clamped composite circular plate is made of an aluminum inner plate ( $\Omega_i$ ) and a steel outer plate ( $\Omega_o$ ) as depicted in figure 4. The inner plate carries a uniformly distributed load of intensity  $q$ , whereas the outer sector is uniformly heated with a temperature differential of  $\Delta T$ . The radii of the inner  $a$  and outer  $b$  sectors are 6 and 12 in.; the thicknesses of  $h_i$  and  $h_o$  are 0.2 and 0.15 in.; the Young's moduli  $E_i$  and  $E_o$  are 10.6 and 30.0 million psi; the Poisson's ratios  $\nu_i$  and  $\nu_o$  are 0.33 and 0.3; and the load  $q$  and temperature differential  $\Delta T$  are 100 psi and 50 °F, respectively.

The classical Beltrami-Michell Formulation cannot be used to solve the problem because the plate has clamped, or displacement boundary conditions, at the outer edge and a mixed condition at the interface. CBMF, however, can solve the mixed boundary value problem. CBMF considers the two moments ( $M_r$  and  $M_\theta$ ) as the primary unknowns. The single displacement function ( $w$ ) is obtained from the moments by backcalculations. The CBMF formulation for the problem is obtained from the stationary condition of the IFM variational functional. Only the final equations are given.

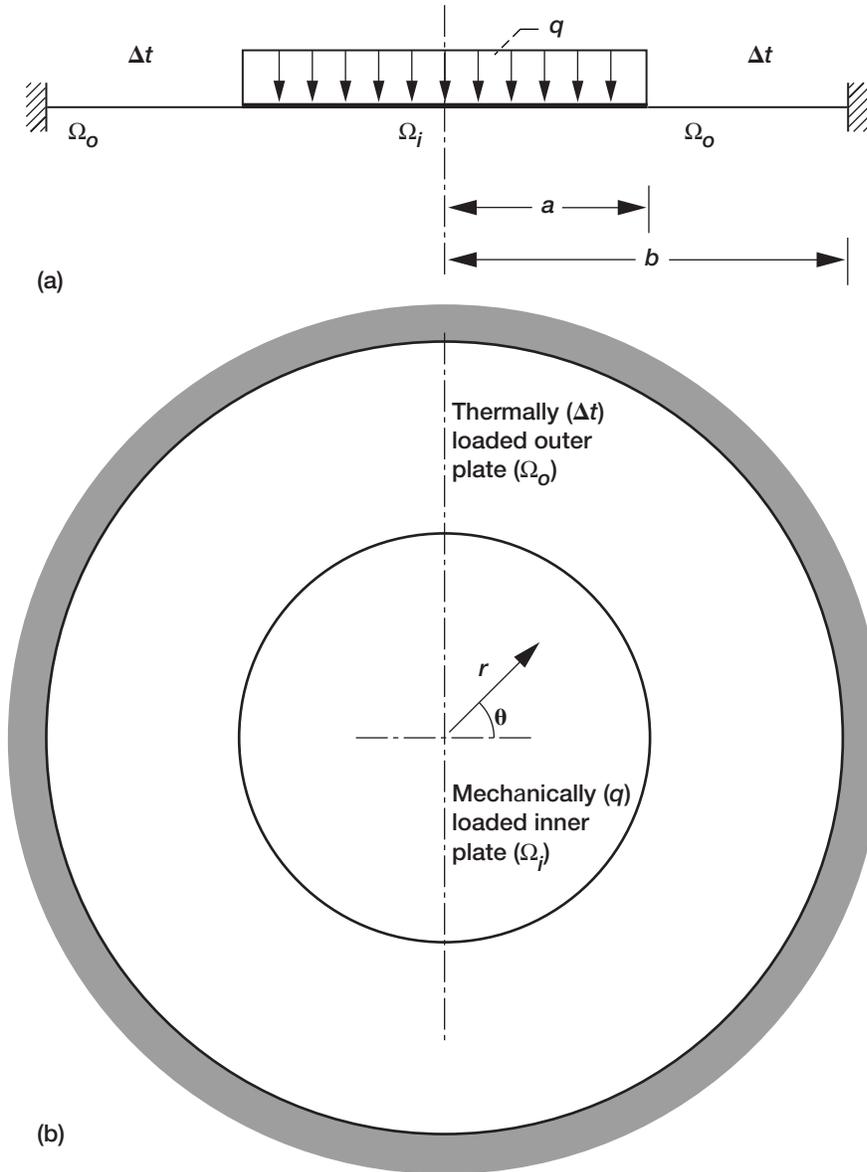


Figure 4.—Composite circular plate subjected to uniform load  $q$  and temperature differential  $\Delta t$ . (a) Side view of plate. (b) Planform of plate.

#### 4.1 Field Equations

In the field the problem has one equilibrium equation and one compatibility condition. The two equations ensure the determinacy of the moment in the field. The field EE is

$$\frac{d^2}{dr^2}(rM_r) - \frac{dM_\phi}{dr} + rq = 0 \quad (19)$$

The field CC is

$$r \frac{d}{dr}(M_\phi - \nu M_r) + (1 + \nu)(M_\phi - M_r) + Kr \frac{a_t}{h} \left( \frac{d\Delta t}{dr} \right) = 0 \quad (20)$$

where the plate rigidity is  $K = Eh^3/[12(1 - \nu^2)]$ .

Field EE (19) and (20) apply to both plate segments ( $\Omega_o$  and  $\Omega_i$ ).

#### 4.2 Boundary Conditions

The problem has three boundary conditions, consisting of two equilibrium equations and one compatibility condition. At the outer clamped boundary the transverse and rotational equilibrium conditions yield two equations:

$$R_w = \frac{d}{dr}(rM_r) - M_\phi \quad (21)$$

$$R_{M_r} = M_r \quad (22)$$

where  $R_w$  and  $R_{M_r}$  represent the transverse and moment reactions. The two boundary equilibrium conditions given by equations (21) and (22) are used to back calculate the two reactions at the outer boundary but are not explicitly used to calculate the moment functions.

The single boundary compatibility condition at the outer boundary has the following form:

$$\frac{1}{K}(M_\phi - \nu M_r) + \alpha_t \left( \frac{\Delta T}{h} \right) = 0 \quad (23 BC_1)$$

This condition, henceforth referred to as  $BC_1$  will be used to determine the moment functions in the CBMF.

At the interface, there are three boundary conditions representing the residue balance condition for equations (21) to (23), discussed earlier. The three conditions at the interface ( $r = a$ ) are

$$M_r^I = M_r^{II} \quad (24 BC_2)$$

$$\frac{d}{dr}(rM_r^I) - M_\phi^I = \frac{d}{dr}(rM_r^{II}) - M_\phi^{II} \quad (25 BC_3)$$

$$\frac{1}{K^I}(M_\phi^{II} - \nu^I M_r^I) + \alpha_t^I \frac{\Delta t}{h^I} = \frac{1}{K^{II}}(M_\phi^{II} - \nu^{II} M_r^{II}) + \alpha_t^{II} \frac{\Delta t}{h^{II}} \quad (26 BC_4)$$

where the superscripts I and II refer to the two sectors at the interface.

The residue in the rotational and transverse equilibrium is accounted for through equations (24) and (25), respectively. The compatibility residue balance is represented by equation (26). The temperature effect is introduced through the BCC, or equation (26). The CBMF will use these three conditions, which will be referred to as  $BC_1$ ,  $BC_2$ ,  $BC_3$ , and  $BC_4$ .

The finite condition of the moment functions ( $M_r$  and  $M_\phi$ ) referred to as ( $BC_5$  and  $BC_6$ ) will also be used in the CBMF analysis. The two field equations and the six boundary conditions are sufficient for the calculation of the two moment functions. In CBMF, displacement is backcalculated from moment functions. This process requires displacement continuity conditions ( $w = 0$  at outer boundary and at the interface:  $w^i = w^0$ ) that are obtained from the stationary condition of the variational functional.

### 4.3. Solution

The field equations (eqs. (19) and (20)) are solved to obtain the following general solutions. For the inner plate,

$$M_r^i(r) = -\frac{B_1}{r^2} + \frac{1}{2} C_1 (1 + \nu_i) \log r + \frac{1}{4} C_1 (1 - \nu_i) + \frac{1}{2} D_1 - \frac{1}{16} (3 + \nu_i) q r^2 \quad (27a)$$

$$M_\phi^i(r) = -\frac{B_1}{r^2} + \frac{1}{2} C_1 (1 + \nu_i) \log r - \frac{1}{4} C_1 (1 - \nu_i) + \frac{1}{2} D_1 - \frac{1}{16} (1 + 3\nu_i) q r^2 \quad (27b)$$

Likewise, the solutions for the outer plate with no distributed load are

$$M_r^o(r) = -\frac{B_2}{r^2} + \frac{1}{2} C_2 (1 + \nu_o) \log r + \frac{1}{4} C_2 (1 - \nu_o) + \frac{1}{2} D_2 \quad (28a)$$

$$M_\phi^o(r) = -\frac{B_2}{r^2} + \frac{1}{2} C_2 (1 + \nu_o) \log r - \frac{1}{4} C_2 (1 - \nu_o) + \frac{1}{2} D_2 \quad (28b)$$

The six integration constants are evaluated for the six boundary conditions for a temperature  $\Delta T$  of 50 °F, a load  $q$  of 100 psi, and the material properties specified earlier. The moment solutions follow.

For the inner plate ( $0 \leq r \leq b$ )

$$M_r^i(r) = 844.05 - 20.81r^2 \quad (29a)$$

$$M_\phi^i(r) = 844.05 - 20.81r^2 \quad (29b)$$

For the outer plate ( $a \leq r \leq b$ ),

$$M_r^o(r) = 2046.63 - \left(5203.06 / r^2\right) - 1170 \log r \quad (30a)$$

$$M_\phi^o(r) = 2676.63 + \left(5203.06 / r^2\right) - 1170 \log r \quad (30b)$$

The moment solutions for the composite plate with the mixed boundary condition are obtained using CBMF without any reference to displacements in the field and on the boundary. Displacements, if required, can be backcalculated from the moments. For this problem, the displacement conditions at the outer boundary ( $w = 0$  at  $r = b$ ) and the continuity conditions at the interface ( $w^i = w^o$ ) are sufficient to calculate the displacement.

For the inner plate, the transverse displacement ( $0 \leq r \leq a$ ) is

$$w^i(r) = 3.1209 - 0.0356r^2 + 0.1757 \times 10^{-3} r^4 \quad (31a)$$

and for the outer plate ( $a \leq r \leq b$ ), it is

$$w^o(r) = 5.3614 - 0.1344r^2 - 0.7296 \log r + 0.04417r^2 \log r \quad (31b)$$

The CBMF solution procedure illustrated here can be extended to analyze displacement and mixed boundary value problems in elasticity as well as for plate and shell problems.

## 5.0 Discussions

The discussion is given under the role of the CC, benefits from the use of the CC, and why the CC weren't formulated earlier.

### 5.1 Role of the Compatibility Conditions

The role of the CC can be perceived from an examination of the stress-strain law, which is universal to all analysis formulations. Hooke's law (see fig. 1), relates stress  $\{\sigma\}$  and strain  $\{\epsilon\}$  through a material matrix  $[\mathfrak{M}]$  as  $(\{\sigma\} = [\mathfrak{M}]\{\epsilon\})$ . The stress must satisfy the state of equilibrium, whereas the strain must satisfy the compatibility. Hooke's law, the equilibrium, and the compatibility are sufficient for determining the stress in an elastic continuum. Conceptually, the calculation of stress can be represented as

$$\left[ \begin{array}{c} \text{Equilibrium Equations} \\ \text{Compatibility Conditions} \end{array} \right] \{\text{Stress}\} = \left[ \begin{array}{c} \text{Mechanical Load} \\ \text{Initial Deformation} \end{array} \right] \quad (32)$$

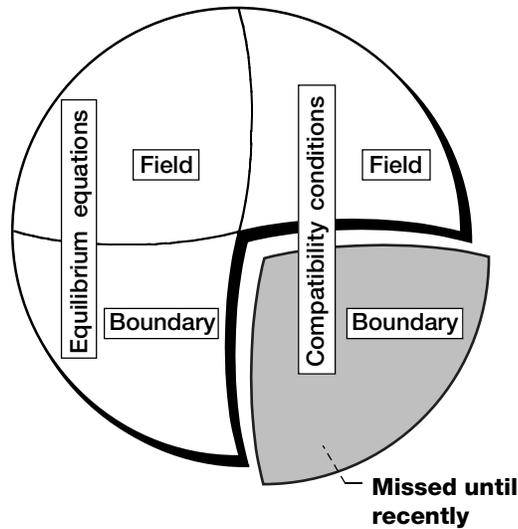


Figure 5.—Equilibrium equations and compatibility conditions in elasticity.

Displacement in explicit terms is not required for the determination of stress. The equilibrium and compatibility concepts of elasticity are depicted in the two halves of the pie diagram in figure 5. The immaturity in the compatibility condition at the boundary is represented by the shaded quarter. In structural mechanics, the compatibility formulated through the concept of redundant forces, “cutting” and “closing” the gap, is not even parallel to the strain formulation of Saint Venant. In other words, the pie diagram also applies to structural mechanics. The theory of elasticity has been developed utilizing the information contained in the three-quarters of the pie diagram. The utilization of the additional quarter should improve the theory making it more robust.

Stress can be determined bypassing the compatibility conditions, but omitting the compatibility can lead to erroneous stress. Even a century ago, this deficiency was observed by Todhunter (ref. 6) while he was scrutinizing astronomer Royal Airy's (1801–1892) attempt to analyze the stress in cantilever and simply supported beams.

“Important Addition and Correction. The solution of the problems suggested in the last two Articles were given—as has already been stated—on the authority of a paper by the late Astronomer Royal, published in a report of the British Association. I now observe, however—when the printing of the articles and engraving of the Figures is already completed—that they cannot be accepted as true solutions, inasmuch as they do not satisfy the general equations (164)

of § 303 [note that the equations in question are the CC]. It is perhaps as well that they should be preserved as a warning to the students against the insidious and comparatively rare error of choosing a solution which satisfies completely all the boundary conditions, without satisfying the fundamental condition of strain [note that the condition in question is the compatibility condition], and which is therefore of course not a solution at all.”

A solution to a problem can be obtained utilizing a subset of elasticity equations. Such a solution was obtained by Airy. Its inaccuracy was pointed out by Todhunter. Strictly speaking, a valid elasticity solution must satisfy the boundary compatibility conditions. It is quite possible that many solutions obtained earlier may satisfy the new conditions without explicit imposition. The “process of evolution” might have eliminated inaccuracy in the traditional solutions because such results have been in existence for over a century. Despite this conciliatory concession, it is but prudent to verify the traditional solution for the compliance of the new boundary conditions.

## 5.2 Benefit in Finite Element Analysis

A finite element method that parallels the completed Beltrami-Michell formulation has been developed. This is referred to as the Integrated Force Method, or IFM. Force parameters are its primal variables (see table I). Displacements are recovered from forces. The IFM equations for a finite element model with  $n$  force and  $m$  displacement unknowns are obtained by coupling the  $m$  equilibrium equations ( $[B]\{F\} = \{P\}$ ) to the  $r = n - m$  compatibility conditions ( $[C][G]\{F\} = \{\delta R\}$ ):

$$\begin{bmatrix} [B] \\ [C][G] \end{bmatrix} \{F\} = \begin{Bmatrix} \{P\} \\ \{\delta R\} \end{Bmatrix} \quad \text{or} \quad [S]\{F\} = \{P^*\} \quad (33)$$

From the internal forces  $\{F\}$  the displacements  $\{X\}$  are backcalculated as

$$\{X\} = [J]\{[G]\{F\} + \{\beta^0\}\} \quad (34)$$

where  $[J] = m$  rows of  $\{[S]^{-1}\}^T$ .

TABLE I.—METHODS OF STRUCTURAL MECHANICS AND ASSOCIATED VARIATIONAL FUNCTIONALS

Method number	Method		Primary variables		Variational functional
	Elasticity	Structures	Elasticity	Structures	
1	Completed Beltrami-Michell Formulation (CBMF)	Integrated Force Method (IFM)	Stresses	Forces	IFM variational functional
2	Airy's formulation	Redundant force method	Stress function	Redundants	Complementary energy
3	Navier's formulation (NF)	Stiffness method (DM)	Displacements	Deflections	Potential energy
4	Hybrid method (HF)	Reissner's method (RM)	Stresses and displacements	Forces and deflections	Reissner's functional
5	Total formulation (TF)	Washizu's method (WM)	Stresses, strains, and displacements	Forces, deformations, and deflections	Washizu's functional

In equations (33) and (34),  $[B]$  is the  $m \times n$  rectangular equilibrium matrix,  $[G]$  is the  $n \times n$  flexibility matrix,  $[C]$  is the  $r \times n$  compatibility matrix,  $\{\delta R\} = -[C]\{\beta^0\}$  is the  $r$ -component effective initial deformation vector,  $\{\beta^0\}$  is the initial deformation vector of dimension  $n$ ,  $[S]$  is the IFM governing matrix, and  $[J]$  is the  $m \times n$  deformation coefficient matrix.

A research-level finite element code, referred to as IFM/Analyzers, has been developed. Results for a few examples obtained using the IFM/Analyzers code and a commercial stiffness method code are depicted in figure 6. The three IFM

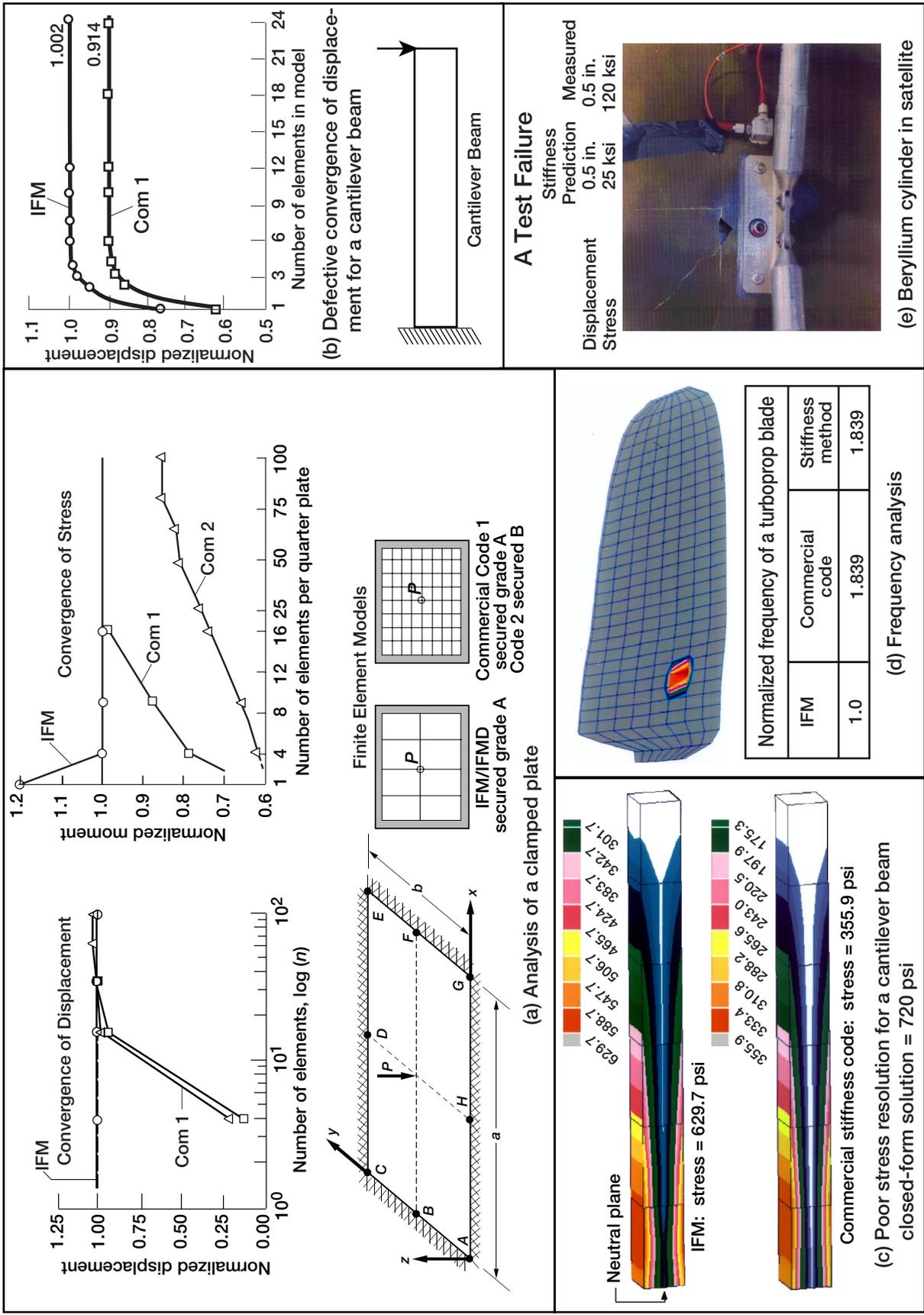


Figure 6.—Solution to a set of examples.

elements used are QUAD0405, which is a four-node membrane element with five force unknowns; HEX2090, which is a 20-node brick element with 90 force unknowns; and PLB0409, which is a four-node plate bending element with nine unknown forces. These are very simple elements that use standard interpolation functions. The elemental generation used numerical integration but did not use reduced integration or bubble function techniques.

Displacement and moment solutions for a flat plate under a concentrated load is given as figure 6(a). A displacement solution to a cantilever beam using membrane elements is shown in figure 6(b). A stress solution to a cantilever beam modeled with brick elements is shown in figure 6(c). The frequency analysis of a turboprop blade using a brick element is shown in figure 6(d) reported under problem 5. The test results and stiffness method predictions are shown in figure 6(e) for a beryllium cylinder. For the examples the integrated force method outperformed the stiffness method, overshadowing the simplicity at its element level. The developer of the commercial code disputed the IFM results (ref. 7). To resolve the dispute, the examples were solved again and a detailed report with several tables was prepared. No numerical error could be detected in any of the example problems. Only one typographical error was found—in the element subtitle. The correct word is “Six/HX20\_90,” not “Three/HX08\_90” as published in the journal (ref. 7). The information submitted to the journal and published in the NASA Technical Memorandum (ref. 8) was correct. The typographical error in the publication is inconsequential to the conclusion.

### 5.3 Why Weren't the Compatibility Conditions Developed Earlier?

Three reasons for the tardy development of the compatibility might be complexity, complacency, and emphasis on solutions. Compatibility conditions in relative terms can be considered to be more complex than other elasticity relations because their formulation requires the variational concept of calculus. If compatibility were simpler, then its derivation would have matured earlier. However, the converse would not be a certainty. In figure 1, we have depicted a partial list of pioneer scientists who contributed to the development of the subject. Complexity cannot be considered to be an insurmountable obstacle with Cauchy, Saint-Venant, Bernoulli, or Maxwell, to mention just a few. Complexity is not the likely reason.

An argument in favor of complacency cannot be justified. Since eminent scientists developed the basic principles of this science, they could not have overlooked the fundamental compatibility concept. In the theory of structure, consider the displacement and classical force methods. Displacement  $\{X\}$  is the unknown in the displacement method  $[K]\{X\} = \{P\}$ . However, the redundant  $\{\mathcal{R}\}$ , not the internal force  $\{F\}$ , is the unknown in the classical force method,  $[H]\{\mathcal{R}\} = \{\mathcal{P}\}$ . Since displacement is the unknown of the displacement method, force should have been the unknown of the force method. In elasticity, Navier's displacement method can solve all three types of boundary value problems. The classical Beltrami-Michell's stress formulation cannot be used to solve the displacement or mixed boundary value problems. Even novice researchers would surely have noticed and questioned the lack of uniformity between the force and displacement methods.

Even an approximate solution can fulfill the need of an industry, which may have little interest in the particular method used. The building and bridge industries required analysis of indeterminate trusses, continuous beams, frames, and other structures. Redundant analysis solved such problems manually. For skeletal frames, the method of moment distributions and Kani's method of stiffness balance were very popular prior to computer automation. Elasticity results, obtained through superposition techniques have served the industry well. The results required by civil, mechanical, and aerospace industries were given primary importance, but the analysis methods used were considered but secondary. The point is that the solutions of engineering problems not only became central to the work but occupied most of the available time of competent researchers, leaving little or no time for them to ponder or address the deficiencies or completion of the theory of compatibility. The engineering problem-solving aspect of structural mechanics was considered to be most important. Such jobs were considered to be glamorous and paid high dividends. But the basic science of structural mechanics, including the completion of the theory of compatibility, appears to have been neglected by industry, research institutions, and academia alike.

Glamour and the many other dividends associated with solving difficult industrial problems are, in our opinion, the primary reasons behind the slow progress of the theory of compatibility. Complexity and complacency may be considered but secondary reasons for neglecting such conditions.

## 6.0 Future Research

The method of force or the completed Beltrami Michell's formulation and the method of displacement or Navier's method are the two primary formulation of elasticity. Other methods are listed in table 1. The boundary compatibility allows facile movement between the five methods given in table 1: (1) CBMF, (2) Airy's stress function method, (3) Navier's displacement method, (4) Reissner's method, and (5) Washizu's formulation. It is prudent to obtain solutions to elasticity problems by both force and displacement methods and to eliminate error by comparison. Research thus far emphasized Navier's displacing method, which might have entered the plateau of diminishing marginal return. The force method has opened up, and researchers should exploit its potential. We propose initial research under two subtopics: completeness of the theory and verification of available solutions.

### 6.1 Completeness of the Theory

We must complete the theory of linear and nonlinear elasticity. We have addressed the theory of linear elasticity. The IFM variational functional has been formulated (ref. 3 and 9), and the BCC have been generated for two- and three-dimensional elasticity problems in Cartesian coordinates. The BCC were also derived for two-dimensional problems in polar coordinates, which has yet to be published. The BCC have been generated for a rectangular plate flexure problem (ref. 10), a circular plate, and a radially symmetrical cylindrical shell (ref. 9). Solutions to plate and cylindrical shell problems have been obtained for mechanical load and temperature distribution. CBMF has been extended to vibration and buckling problems utilizing the concept of "stress mode shapes" (ref. 11). In addition, the noncompliance of the CC at the boundary has been shown for a simple elasticity problem (ref. 11). CBMF needs to be developed in curvilinear coordinates for three-dimensional elasticity. The procedure should be extended to plates and shells. Since there are many different geometrical shell configurations (like cylinder, spherical, conical, and paraboloidal shapes), the feasibility of training the IFM variational functional through a computer program using symbolic language should be explored. The linear CBMF analysis should be extended next to finite deformation elasticity theory in Lagrangian and Eulerian coordinates. The steps followed for linear elasticity can be adopted for nonlinear problems: an extension of the variational functional including stress functions, variational operations, recovering equations, and their interpretation and attributes. Prior to three-dimensional nonlinear analysis, the basic steps can be exercised for a two-dimensional model.

### 6.2 Verification of Available Solutions

Solutions are available to many elasticity problems.. We must verify these solutions for the compliance of the BCC. This exercise might be trivial for some problems, but new solutions might be required for others. This exercise cannot be avoided because classical solutions are still in use to verify numerical solutions to engineering problems. For three-dimensional elasticity, the compatibility conditions on a boundary surface will be satisfied provided the following curvature terms vanish:

$$\begin{aligned}\frac{\partial^2 u}{\partial y^2} &= \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial z \partial x} = \frac{\partial^2 u}{\partial x \partial y} = 0 \\ \frac{\partial^2 v}{\partial y^2} &= \frac{\partial^2 v}{\partial z^2} = \frac{\partial^2 v}{\partial y \partial z} = \frac{\partial^2 v}{\partial x \partial y} = 0 \\ \frac{\partial^2 w}{\partial x^2} &= \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial y \partial z} = \frac{\partial^2 w}{\partial z \partial x} = 0\end{aligned}\tag{35}$$

The boundary compatibility conditions can be factorized to obtain:

$$\begin{aligned}
\left( a_{vy} \frac{\partial}{\partial z} - a_{vz} \frac{\partial}{\partial y} \right) \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) &= 0 \\
\left( a_{vx} \frac{\partial}{\partial x} - a_{vx} \frac{\partial}{\partial z} \right) \left( \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) &= 0 \\
\left( a_{vx} \frac{\partial}{\partial y} - a_{vy} \frac{\partial}{\partial x} \right) \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) &= 0
\end{aligned} \tag{36}$$

Boundary compatibility compliance requires equality of the following rotation terms:

$$\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z} \quad \frac{\partial w}{\partial x} = \frac{\partial u}{\partial z} \quad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \tag{37}$$

The displacement  $w$  with respect to  $y$  (or in the  $v$ -displacement direction), must be equal to the derivative of the displacement  $v$  with the respect to  $z$  (or  $w$ -displacement direction), and so forth.

## 7.0 Conclusions

A deficiency in the theory of elasticity has been eliminated through the formulation of the compatibility conditions on the boundary of an elastic continuum. These equations have completed the stress formulation in elasticity—a task that was originally attempted by Beltrami and Michell at the end of the 19th century. This primal method of stress can handle all three classes of boundary value problems for static, thermal, and dynamic loads. Elasticity solutions generated by Navier’s displacement method and Airy’s stress function formulation should be verified for the compliance of the novel compatibility conditions because these equations expressed in displacements do not become trivial functions like the field compatibility conditions. Structural analyses and design can benefit from the use of the stress formulation. The compability conditions remain to be formulated in nonlinear elasticity.

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<b>REPORT DOCUMENTATION PAGE</b>			<i>Form Approved</i> <i>OMB No. 0704-0188</i>	
Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188), Washington, DC 20503.				
<b>1. AGENCY USE ONLY</b> <i>(Leave blank)</i>	<b>2. REPORT DATE</b> September 2001	<b>3. REPORT TYPE AND DATES COVERED</b> Technical Paper		
<b>4. TITLE AND SUBTITLE</b>  Stress Formulation in Three-Dimensional Elasticity			<b>5. FUNDING NUMBERS</b>  WU-505-63-5B-00	
<b>6. AUTHOR(S)</b>  Surya N. Patnaik and Dale A. Hopkins				
<b>7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)</b>  National Aeronautics and Space Administration John H. Glenn Research Center at Lewis Field Cleveland, Ohio 44135-3191			<b>8. PERFORMING ORGANIZATION REPORT NUMBER</b>  E-10106-1	
<b>9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)</b>  National Aeronautics and Space Administration Washington, DC 20546-0001			<b>10. SPONSORING/MONITORING AGENCY REPORT NUMBER</b>  NASA TP-2001-210515	
<b>11. SUPPLEMENTARY NOTES</b>  Surya N. Patnaik, Ohio Aerospace Institute, 22800 Cedar Point Road, Brook Park, Ohio 44142; and Dale A. Hopkins, NASA Glenn Research Center. Responsible person, Dale A. Hopkins, organization code 5930, 216-433-3260.				
<b>12a. DISTRIBUTION/AVAILABILITY STATEMENT</b>  Unclassified - Unlimited Subject Category: 39 Available electronically at <a href="http://gltrs.grc.nasa.gov/GLTRS">http://gltrs.grc.nasa.gov/GLTRS</a> This publication is available from the NASA Center for AeroSpace Information, 301-621-0390.			<b>12b. DISTRIBUTION CODE</b>	
<b>13. ABSTRACT</b> <i>(Maximum 200 words)</i>  The theory of elasticity evolved over centuries through the contributions of eminent scientists like Cauchy, Navier, Hooke Saint Venant, and others. It was deemed complete when Saint Venant provided the strain formulation in 1860. However, unlike Cauchy, who addressed equilibrium in the field and on the boundary, the strain formulation was confined only to the field. Saint Venant overlooked the compatibility on the boundary. Because of this deficiency, a direct stress formulation could not be developed. Stress with traditional methods must be recovered by backcalculation: differentiating either the displacement or the stress function. We have addressed the compatibility on the boundary. Augmentation of these conditions has completed the stress formulation in elasticity, opening up a way for a direct determination of stress without the intermediate step of calculating the displacement or the stress function. This Completed Beltrami-Michell Formulation (CBMF) can be specialized to derive the traditional methods, but the reverse is not possible. Elasticity solutions must be verified for the compliance of the new equation because the boundary compatibility conditions expressed in terms of displacement are not trivially satisfied. This paper presents the variational derivation of the stress formulation, illustrates the method, examines attributes and benefits, and outlines the future course of research.				
<b>14. SUBJECT TERMS</b> Boundary compatibility; Variational functional; Beltrami-Michell; Integrated Force Method			<b>15. NUMBER OF PAGES</b> 25	
			<b>16. PRICE CODE</b>	
<b>17. SECURITY CLASSIFICATION OF REPORT</b> Unclassified	<b>18. SECURITY CLASSIFICATION OF THIS PAGE</b> Unclassified	<b>19. SECURITY CLASSIFICATION OF ABSTRACT</b> Unclassified	<b>20. LIMITATION OF ABSTRACT</b>	